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ON THE STABILITY OF UNSTEADY MOTIONS*

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A generalization of the fundamental Lyapunov and Chetayev theorems in which the requirement of the existence of an infinitely small upper limit for the Lyapunov function is relaxed, is presented. The generalized theorems prove that it is sufficient for Lyapunov's function either to admit of the infinitely small upper limit, or to be bounded with respect to time t in certain time intervals t_i for which the condition $\sum_i t_i \to \infty$ holds as $t \to \infty$.

Theorems on stability and instability are formulated to a first approximation. The application of the theorems is illustrated by the example of the stability of a second-order system, with the construction of a Lyapunov generalized function.

1. The theory of stability in the second Lyapunov method has recently been fairly widely developed, and the Lyapunov theorems have been generalized, but in all the theorems and their generalizations, see /1—7/, the requirement of the existence of an infinitely small upper limit of the Lyapunov function remains unchanged. However, this requirement is very severe, and it can be relaxed. Consider the following equations of the perturbed motion:

$$dx_s/dt = f_s(t, \mathbf{x}),\ (\mathbf{x} = (x_1, \ldots, x_n))\ (s = 1, \ldots, n) \tag{1.1}$$

As regards the right-hand sides of these equations, we will assume that in the domain

$$|x_s| < \varepsilon,\quad t \geqslant t_0 > 0 \tag{1.2}$$

they are continuous and admit of the existence of a unique solution for the specified initial conditions, and that the conditions $f_s(t, 0) = 0$ are satisfied; here ε is a positive constant. In the same domain we shall consider functions $V(t, \mathbf{x})$ which, we assume, have continuous partial derivatives and vanish when $\mathbf{x} = 0$.

Let us introduce the following definitions.

Definition 1. We shall describe the function $V(t, \mathbf{x})$ as positive definite (negative definite) in the broad sense if in the domain $|x_s| < \varepsilon$ as small as desired, and for t as large as desired (say $t \in T_v$) there exist the time intervals t_{v1}^i such that in them the function V satisfies the condition $V \geqslant W(\mathbf{x})$, $(V \leqslant -W(\mathbf{x}))$ here W is a positive definite function independent of t. In the same domain $|x_s| < \varepsilon$ and for $t \in T_v$ time intervals t_{v2}^i can exist in which the function V satisfies the condition $V(t, \mathbf{x}) \geqslant 0$, $(V \leqslant 0)$. The intervals t_{v1}^i and t_{v2}^i occupy T_v totally.

Definition 2. We shall describe the function $V(t, \mathbf{x})$ as a function which conditionally admits of an infinitely small bound if in the domain $|x_s| < \varepsilon$, as small as desired, and for $t \in T_v$ time intervals t_{v3}^i exist in which the function V tends to zero as $|\mathbf{x}| \to 0$ uniformly in t. In the same domain $|x_s| < \varepsilon$ and for $t \in T_v$ time intervals t_{v4}^i can exist in which the function V may not admit of an infinitely small upper limit. The time intervals t_{v3}^i and t_{v4}^i occupy the whole of T_v.

2. Taking into account the above definitions, we can formulate the following generalizations of the basic Lyapunov theorems.

Theorem 1 (on the asymptotic stability). Suppose that for the differential equations of perturbed motion one can find a fixed-sign function $V(t, \mathbf{x})$ which admits of an infintely small upper bound, and whose total derivative dV/dt, compiled on the strength of Eq.(1.1), is a function of fixed sign in the broad sense, opposite to that of V. If at the same time $\sum_i t_{v1}^i \to \infty$ as $t \to \infty$ and all the intervals t_{v1}^i are fully contained in t_{v3}^i, then the unperturbed motion is asymptotically stable.

Proof. Assume that $V(t, \mathbf{x})$ is a positive definite function; then in the domain (1.2) the inequality

$$V(t, \mathbf{x}) \geqslant W(\mathbf{x}) \tag{2.1}$$

holds. Here W is a certain negative definite function independent of t.

Moreover, if in the same domain dV/dt there is a negative definite function in the broad snese, then

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$$dV/dt \leqslant W_1(\mathrm{x}),\ t \in t^i_{v\cdot 1};\ dV/dt \leqslant 0,\ t \in t^i_{v\cdot 2} \qquad (2.2)$$

where W_1 is a certain positive-definite function independent of t.

We shall regard the quantities x_s as time functions which satisfy the differential equations of the perturbed motion (1.1), assuming that the initial values of $x_s(t_0)$ satisfy conditions (1.2). Since an unperturbed motion is steady in any case, the quantity ε can be chosen to be so small that for all $t \geqslant t_0 > 0$ the quantities x_s remain in the domain (1.2). But then the derivative dV/dt will be non-positive, and consequently when t increases without limit the function V will tend to a certain limit, all the time remaining greater than this limit.

We shall show that the limit in question equals zero. Let us assume the opposite, namely that the limit equals a certain positive quantity $\alpha \neq 0$, that is the inequality

$$V(t,\ \mathrm{x}) > \alpha \qquad (2.3)$$

holds for all $t \geqslant t_0 > 0$.

However, since in the intervals $t^i_{v\cdot 1}$ the function V admits of an infinitely small upper limit, the inequality

$$X(t) = \max\{|x_1(t)|, \ldots, |x_n(t)|\} \geqslant \lambda,\ t \in t^i_{v\cdot 1} \qquad (2.4)$$

where λ is a certain fairly small positive number, will be satisfied.

But if the above inequality is satisfied for $t \in t^i_{v\cdot 1}$, so is the inequality

$$dV/dt \leqslant -l,\quad t \in t^i_{v\cdot 1};\ dV/dt \leqslant 0,\ t \in t^i_{v\cdot 2}$$

where l is a positive number different from zero, which is the exact lower limit of the function dV/dt when $t \in t^i_{v\cdot 1}$, and when condition (2.3) is satisfied.

Therefore, for all $t \geqslant t_0 > 0$

$$V(t, \mathrm{x}) = V(t_0, \mathrm{x}^\circ) + \int_{t_0}^{t} \frac{dv}{dt}(t, \mathrm{x})\, dt \leqslant V(t_0, \mathrm{x}^\circ) - l \sum_i t^i_{v\cdot 1}$$

i.e. as $t \to \infty$ the right-hand side tends to $-\infty$. This contradicts (2.3), hence

$$\lim_{t\to\infty} V(t,\ \mathrm{x}) = 0$$

Consequently, this holds for the function $W(\mathrm{x})$ as well; then

$$\lim_{t\to\infty} X(t) = \lim_{t\to\infty} \max\{|x_1(t)|, \ldots, |x_n(t)|\} = 0$$

which proves the theorem.

Theorem 2 (on instability). Suppose that a function $V(t, \mathrm{x})$ exists which admits of an infinitely small upper limit, and that its time derivative compiled on the strength of the equations of perturbed motion is a function with a fixed sign in the broad sense. The function V itself can, for values of x_s as small as desired, and for values of t as large as desired, take values of the same sign as that of the derivative. If at the same time $\sum_i t^i_{v\cdot 1} \to \infty$ as $t \to \infty$, and $t_{v2}{}^i$ contain all intervals $t^i_{v\cdot 1}$, then the undisturbed motion is unstable.

Proof. We assume that dV/dt is positive definite in the broad sense, that is in the domain (1.2) the equations

$$dV/dt \geqslant W_1(\mathrm{x}),\ t \in t^i_{v\cdot 1};\ dV/dt \geqslant 0,\ t \in t^i_{v\cdot 2} \qquad (2.5)$$

where $W_1(\mathrm{x})$ is a positive definite function independent of t, hold.

Let us consider the solution $x_s = x_s(t)$ of the equations of motion for which the initial values $x_s{}^\circ = x_s(t_0)$ are chosen from the condition

$$|x_s{}^\circ| \leqslant \eta,\quad V(t_0,\ \mathrm{x}^\circ) > 0$$

where η is a fairly small positive number.

We shall show that this solution will certainly at some instant of time leave the domain (1.2). Let us assume that the solution all the time remains in this domain. On the basis of (2.5) the derivative of the function V is in any case negative, therefore

$$V(t,\ \mathrm{x}) \geqslant V(t_0,\ \mathrm{x}^\circ) \qquad (2.6)$$

However, if condition (2.6) is satisfied, and in the intervals $t \in t^i_{v\cdot 1}$ the function V admits of an infinitely small upper limit, the inequality

$$X(t) = \max\{|x_1(t)|, \ldots, |x_n(t)|\} \geqslant \lambda,\ t \in t^i_{v\cdot 1} \qquad (2.7)$$

where λ is a fairly small positive number, holds.

Then it follows from (2.5) that

$$dV/dt \geqslant l,\ t \in t^i_{v\cdot 1} \qquad (2.8)$$

where l is a positive number different from zero, which is the lower limit of the function

dV/dt when $t \in t^i_{v\cdot 1}$, and (2.7) holds.

Taking into account condition (2.8), we obtain

$$V(t, \mathbf{x}) = V(t_0, \mathbf{x}^\circ) + \int_{t_0}^{t} \frac{dV}{dt}(t, \mathbf{x})\,dt \geqslant V(t_0, \mathbf{x}^\circ) + l \sum_i t^i_{v\cdot 1} \tag{2.9}$$

However, satisfaction of (2.9) is impossible since the function V, which conditionally admits of an infinitely small upper limit when $t \in t^i_{v\cdot 1}$, is at least limited, and the right-hand side of (2.9) tends to infinity as $t \to \infty$

It follows from this contradiction that the solution $x_s = x_s(t)$ will certainly at some instant of time leave the domain (1.2) which is independent of the initial values, and since these values are as small as desired, the unperturbed motion is unstable.

To generalize the Chetayev theorem on instability, we shall consider the neighbourhood of the origin of coordinates for the space of variables $x_1, \ldots, x_n$, bounded by the surface $V = 0$ in which the function V takes positive values. We shall refer to this neighbourhood as the domain $V > 0$. Let us assume that the function V has the following properties.

1°. For t as large as desired, and in a neighbourhood of the origin as small as desired, a domain $V > 0$ exists.

2°. In the domain $V > 0$, for $t_{v3}{}^i$ the function V is bounded, and for $t_{v4}{}^i$ it can also be unbounded.

3°. In the domain $V > 0$, for all t and $\mathbf{x}$ connected by the relation

$$V(t, \mathbf{x}) > \alpha \tag{2.10}$$

where α is a positive number as small as desired, the inequality

$$dV/dt \geqslant l, \ t \in t^i_{v\cdot 1}; \ dV/dt \geqslant 0, \ t \in t^i_{v\cdot 2} \tag{2.11}$$

where l is a certain positive number depending on α, is satisfied.

Theorem 3 (on instability). If for the differential functions of perturbed motion there is a function which satisfies Conditions 1°– 3°, and $\sum_i t^i_{v\cdot 1} \to \infty$ as $t \to \infty$, then the unperturbed motion is unstable.

Proof. Let us set a fairly small neighbourhood of the origin, which contains the domain $V > 0$. Consider the solution $x_s = x_s(t)$ of the equation of perturbed motion with the initial values $x_s{}^\circ = x(t_0)$ numerically selected as small as desired such that $V(t_0, \mathbf{x}^\circ) > \alpha$.

For $V > \alpha$, the derivative dV/dt is non-negative, therefore the function $V(t, x(t))$ is non-decreasing and, consequently, the quantities $x_s(t)$ remain in the domain $V > \alpha$, at least as long as inequality (1.2) remains true.

We assume that (1.2) is never violated, but for $t > t_0$ the condition

$$V(t, \mathbf{x}(t)) \geqslant V(t_0, \mathbf{x}^\circ)$$

is satisfied since dV/dt at any rate, non-negative, Hence it follows that

$$dV/dt \geqslant l, \ t \in t^i_{v\cdot 1}; \ dV/dt \geqslant 0, \ t \in t^i_{v\cdot 2}$$

We then obtain

$$V(t, \mathbf{x}(t)) \geqslant V(t_0, \mathbf{x}^\circ) + l \sum_i t^i_{v\cdot 1}$$

But this is impossible because in the domain $t \in t^i_{v\cdot 1}$ the function V is bounded.

Thus, at some instant of time the solution will certainly leave the domain (1.2), and since the quantities $x_s(t_0)$ can be taken as small as desired, the unperturbed motion is unstable.

Considering the theorems above, we can generalize the theorems on stability and instability to a first approximation. Consider the system

$$dx_s/dt = \sum_{i=1}^{n} p_{is}(t)\,x_i + \vartheta_s(t, x) \quad (s = 1, \ldots, n) \tag{2.12}$$

where p_{is} are arbitrary, continuous and bounded time functions when $t \geqslant t_0 > 0$ and ϑ_s are functions which in the domain (1.2) satisfy the inequalities

$$|v_s(t, \mathbf{x})| \leqslant A \sum_{i=1}^{n} x_i^2 \tag{2.13}$$

(A is a positive constant).

Consider, together with (2.12), the first-approximation system,

$$\frac{dx_s}{dt} = \sum_{i=1}^{n} p_{is}(t)\,x_i \quad (s = 1, \ldots, n) \tag{2.14}$$

Among the Lyapunov functions we shall investigate the quadratic forms of the variables $x_1, \ldots, x_n$, of the form

$$V = \varphi(t) \sum_{i,j=1}^{n} a_{ij}(t) x_i x_j \tag{2.15}$$

Here $a_{ij}(t)$ are arbitrary and continuous time functions bounded for $t \geqslant t_0 > 0$, and $\varphi(t)$ is a continuous, fixed-sign and semibounded time function; i.e. it is bounded for $t_{\varphi 3}{}^i$, and may be unbounded for $t_{\varphi 4}{}^i$.

Theorem 4. Suppose that for the set of Eqs.(2.14) of the first-approximation one can find a fixed-sign quadratic form of type (2.15) for which the time derivative dV/dt, constructed on the strength of Eqs.(2.14), is the qudratic fixed-sign form of sign opposite to that of V,

$$\frac{dV}{dt} = \varphi(t) \sum_{i,j=1}^{n} b_{ij}(t) x_i x_j \tag{2.16}$$

where $b_{ij}(t)$ are continuous time functions bounded for $t \geqslant t_0 > 0$. If, in addition, $\sum_i t^i_{\varphi 3} \to \infty$ is satisfied for the function $\varphi(t)$ as $t \to \infty$, then the unperturbed motion for Eqs.(2.12) is asymptotically stable for any choice of the functions ϑ_s which satisfy inequalities (2.13).

Proof. We assume that the function V is positive definite. On the strength of Eqs.(2.12) the total derivative of this function can be expressed as

$$\frac{dV}{dt} = \varphi(t) \Big[\sum_{i,j=1}^{n} b_{ij}(t) x_i x_j + u \Big] \tag{2.17}$$

Taking (2.12) and (2.17) into account we have

$$u = \sum_{i,j,s=1}^{n} a_{ijs}(t) \frac{\partial (x_i x_j)}{\partial x_s} \vartheta_s \leqslant B \sum_{i=1}^{n} |x_i{}^3| \tag{2.18}$$

(here B is a positive constant).

The expansion of the function u starts with terms of order no less than the third; therefore function (2.17) is negative definite irrespective of the choice of the function ϑ_s. Therefore, the function V satisfies all the conditions of the generalized Theorem 1, and the unperturbed motion of system (2.12) is asymptotically stable.

Theorem 5. Suppose that for the system of Eqs.(2.14) to a first approximation one can find a quadratic form (2.15) for which the time derivative dV/dt, compiled on the strength of (2.14), is a fixed-sign quadratic form, and can be expressed as (2.16). The quadratic form (2.15) can take the value of the same sign as that of the derivative. If, in addition, for the function $\varphi(t)$ $\sum_i t^i_{\varphi 3} \to \infty$ is satisfied as $t \to \infty$, then the unperturbed motion for Eq. (2.12) is unstable for any choice of functions ϑ_s which satisfy inequalities (2.13).

Proof. The total derivative of the function V for system (2.12) can be written in the form (2.17). The derivative is of fixed sign, and regarding the sign it agrees with (2.16) because conditions (2.18) are satisfied, and the function V itself may take the values of the same sign as the derivative. Consequently, the function V satisfies all the conditions of the generalized Theorem 2, and the undisturbed motion of system (2.12) is unstable.

3. As an example of the application of the generalized theorems, we shall consider the second-order system introduced by Perron, /8/, and discussed in /3/,

$$\begin{aligned} x^{\cdot} &= -ax + \varphi_1(t, x, y) \\ y^{\cdot} &= [\sin \ln (t+1) + \cos \ln (t+1) - 2a]\, y + \varphi_2(t, x, y) \end{aligned} \tag{3.1}$$

where $a > 0.5$: the functions $\varphi_1(t, x, y)$, $\varphi_2(t, x, y)$ satisfy the conditions

$$|\varphi_i(t, x, y)| \leqslant A(x^2 + y^2) \quad (i = 1, 2) \tag{3.2}$$

For the first-approximation system

$$x^{\cdot} = -ax, \quad y^{\cdot} = [\sin \ln (t+1) + \cos \ln (t+1) - 2a]\, y \tag{3.3}$$

one can construct the function

$$V = x^2 + y^2/2a - 1 \exp \{2(t+1)[1 - \sin \ln (t+1)]\} \tag{3.4}$$

which conditionally admits of an infinitely small upper limit. The total derivative of this function,

$$dV/dt = -2ax^2 - 2y^2 \exp \{2(t+1)[1 - \sin \ln (t+1)]\} \tag{3.5}$$

is negative definite, i.e., the conditions of Theorem 1 are satisfied. Consequently, the unperturbed motion of system (3.3) is asymptotically stable.

However, in constructing function (3.4), the conditions of Theorem 4 were not taken into account. For this reason we can say nothing about the nature of the unperturbed motion of system (3.1) because the function (3.4) cannot any longer be taken as a Lyapunov function of system (3.1). For example, if we assume that

$$\varphi_1 = y^2, \ \varphi_2 = 0 \tag{3.6}$$

then the total derivative for function (3.4), compiled on the strength of system (3.1), taking into account (3.6),

$$dV/dt = -2ax^2 - [2y^2 - 2x^2y/2a - 1] \exp\{2(t+1)\ [1 - \sin\ln(t+1)]\} \tag{3.7}$$

is alternating for sufficiently large t.

We can construct for system (3.1) a Lyapunov function which satisfies the conditions of Theorem 4 in the form

$$V = (x^2 + y^2)\,\Gamma(t), \ \ \Gamma(t) = \exp_{t_0}\int_{t_0}^{t} \gamma(t)\,dt \tag{3.8}$$

$$\gamma(t) = \begin{cases} 2a - \delta_1, \ 2a < 2a - \sin\ln(t+1) - \cos\ln(t+1) \\ 2[2a - \sin\ln(t+1) - \cos\ln(t+1)] - \delta_2, \\ 2a \geqslant 2a - \sin\ln(t+1) - \cos\ln(t+1) \end{cases}$$

(here δ_1 and δ_2 are small positive numbers).

On the strength of system (3.1), the total derivative can be written as

$$\frac{dV}{dt} = -\{(2a - \gamma)x^2 + [4a - 2\sin\ln(t+1) - 2\cos\ln(t+1) - \gamma]\,y^2 +$$
$$2x\varphi_1 + 2y\varphi_2\}\,\Gamma(t) \leqslant -\{\delta(x^2 + y^2) + B[|x^3| + |y^3|]\}\,\Gamma(t)$$

where B is a certain constant, and δ is less than δ_1 or δ_2.

To satisfy the conditions of Theorem 4, we must find a coefficient a such that the function $\Gamma(t)$ is semibounded for $t \geqslant t_0 > 0$ When determining this coefficient there is no need to calculate the function $\Gamma(t)$ in the interval $t_0 - \infty$, and it is sufficient to find the function in the interval $t_0 - t_0 + \exp 2\pi$, where t_0 is chosen from the condition

$$\sin\ln(t+1) + \cos\ln(t+1) = 2a$$

In this case we can show that if

$$\Gamma(t_0 + e^{2\pi}) > 0, \quad \text{then also} \quad \Gamma(\infty) > 0 \tag{3.9}$$

and if

$$\Gamma(t_0 + e^{2\pi}) < 0, \quad \text{then also} \quad \Gamma(\infty) < 0 \tag{3.10}$$

However, if condition (3.9) is satisfied we can always find a conditionally admissible infinitely small upper limit of Lyapunov's function (3.8) and, consequently, satisfy the conditions of Theorem 4.

Thus, when conditions (3.9) is satisfied, which, as approximate calculations show, holds at least for $a \geqslant 0.574$ (for $a \leqslant 0.573$ conditions (3.9) already cease to be satisfied); the unperturbed motion of the complete system (3.1) will also be asymptotically stable for any choice of the functions $\varphi_1(t, x, y), \varphi_2(t, x, y)$, which satisfy condition (3.2), all the conditions of Theorem 4 being satisfied.

If we turn to the Persidskii criterion, see /9/, satisfaction of the inequalities

$$X_{ij}(t, t_0) < B \exp[\alpha(t - t_0)] \tag{3.11}$$

is necessary. Here $x_{ij}(t, t_0)$ is the fundamental system of solutions (3.3), and B and α are positive constants independent of t_0. From the approximate estimate of expression (3.11) written for Eqs.(3.3), that is from the expressions

$$y(t, t_0) = \exp[(t+1)\ \sin\ln(t+1) - 2at - (t_0+1)\ \sin\ln(t_0+1) + 2at_0]$$

for $2a \leqslant 1.39$, $t + 1 = \exp(2\pi m + \pi/3)$, $t_0 + 1 = \exp(2\pi m + \pi/6)$ we have

$$y(t, t_0) \leqslant \exp[0.005 \exp(2\pi m + \pi/6)] \tag{3.12}$$

This means that the Persidskii conditions are not satisfied since as m increases the right-hand side of (3.12) increases without limit. Consequently, for $2a \leqslant 1.39$ the equivalent theorems of Malkin and Perron (see /3, 8/) will be invalid. Also, the Lyapunov criterion is not satisfied since system (3.1) is incorrect. The conditions of Malkin's theorem can be satisfied for $2a > \sqrt{2}$ since it is sufficient to take the Lyapunov function $V = \frac{1}{2}(x^2 + y^2)$ whose total derivative is

$$dV/dt = -ax^2 + [\sin\ln(t+1) + \cos\ln(t+1) - 2a]\,y^2$$

Thus, the use of the generalized theorems enables wider boundaries of the stability domain of unsteady motions to be found and, therefore, enables certain stability problems of such motions, which were outside the scope of previously-known theorems to be solved.

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NON-LOCAL CRITERIA FOR THE EXISTENCE AND STABILITY OF PERIODIC OSCILLATIONS IN AUTONOMOUS HAMILTONIAN SYSTEMS*

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The conditions under which single-parameter families of periodic solutions (the existence in a sufficiently small neighbourhood of the origin of coordinates follows from the Lyapunov theorem (see /1/)) can be continued in a parameter to the boundary of the given domain, in particular to a certain isoenergetic surface, are found. These conditions, which can be verified by the use of the Hessian of a Hamilton function, also ensure the orbital stability of solutions to a first approximation. Bilateral estimates of the oscillation periods are obtained, and it is established that any solution with a period which satisfies such an estimate belongs to the corresponding family. As an example, the non-linear oscillations of a string with lumped masses are examined.

The well-known non-local results relevant to the periodic oscillations of autonomous Hamiltonian systems are, as a rule, theorems on the existence of periodic solutions (see reviews /2—4/). One group of papers establishes the existence of periodic solutions with a specified value of the Hamiltonian, and other papers, establish solutions with a specified period; in the first case assumptions and made regarding the form of the corresponding constant energy surface; and in the second assumptions are made regarding the behaviour of the Hamiltonian in the vicinity of the equilibrium configuration and at infinity. The majority of the results were obtained by variational methods, the desired periodic solutions being identified with the stationary points of certain functionals. The discussion in the present paper is based on other concepts.

1. Consider the system.

$$x_i^{\cdot}=\frac{\partial H}{\partial x_{i+n}},\quad x_{i+n}^{\cdot}=-\frac{\partial H}{\partial x_i},\quad i=1,\ldots,n \tag{1.1}$$

where $x_1,\ldots,x_n$ and $x_{n+1},\ldots,x_{2n}$ are the generalized coordinates and momenta, and $H(x_1,\ldots,x_{2n})$ is the Hamiltonian function, doubly differentiable with respect to x_i.

Let $\mathbf{x}^\circ(t)=(x_1^\circ(t),\ldots,x_{2n}^\circ(t))'$ be a periodic solution of system (1.1) with period T_0 (here the prime denotes transposition). The corresponding variational equation is

$$J\mathbf{y}^{\cdot}=A_0(t)\,\mathbf{y} \tag{1.2}$$

$$A_0(t)=\|a_{ik}^\circ(t)\|_1^{2n},\quad a_{ik}^\circ=\left.\frac{\partial^2 H}{\partial x_i\partial x_k}\right|_{\mathbf{x}=\mathbf{x}^\circ(t)}$$

$$J=\begin{Vmatrix}0 & -I_n\\ I_n & 0\end{Vmatrix},\quad \mathbf{y}=(y_1,\ldots,y_{2n})'$$

where I_n denotes the unit matrix of order n.

We will recall some well-known facts. System (1.1) admits of the integral

$$H(x_1(t),\ldots,x_{2n}(t))=\text{const}, \tag{1.3}$$

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